SINP-TNP/97-08
hep-th/9706165

SL(2, Z) Multiplets of Type II Superstrings in $D < 10$

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ABSTRACT

It has been shown recently that the toroidally compactified type IIB string effective action possesses an SL(2, R) invariance. Using this symmetry we construct an infinite family of macroscopic string-like solutions permuted by SL(2, Z) group for type II superstrings in $4 \leq D < 10$. These solutions, which formally look very similar to the corresponding solutions in $D = 10$, are characterized by two relatively prime integers corresponding to the 'electric' charges associated with the two antisymmetric tensor fields of the strings. Stability of these solutions is discussed briefly in the light of charge conservation and the tension gap equation.

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It is well-known that the equations of motion of type IIB supergravity theory are invariant under an SL(2, R) group known as the supergravity duality group [1]. A discrete subgroup of this group has been conjectured to be the exact symmetry of the full quantum type IIB string theory [2]. As the string theory coupling constant transforms non-trivially under this SL(2, R) transformation [3,4,5], this symmetry is non-perturbative. So, in general it is not possible to prove this conjecture in the perturbative framework of string theory. A strong evidence in favor of this conjecture has been given by Schwarz [4] when he showed that certain BPS saturated macroscopic string-like solutions of type IIB string theory form an SL(2, Z) multiplet. These solutions, when characterized by two relatively prime integers corresponding to the charges associated with the two antisymmetric gauge fields (from NS-NS and R-R sectors), are stable and do not decay further into strings with lower charges [6]. The tensions as well as the charges associated with the strings have been shown to be given by SL(2, Z) covariant expressions.

It has been shown recently at the level of low energy effective action that this SL(2, R) invariance of the type IIB theory survives the toroidal compactification [7,8]. In fact, this is not surprising since a symmetry in a higher dimensional theory should become a part of the bigger symmetry in the lower dimensional theory, although in this case, it requires quite non-trivial calculation to prove the invariance. So, eventhough SL(2, R) remains a symmetry group of type II theory in $D < 10$, the whole symmetry group, known as the U-duality group [1,2] (E_9, E_8, E_7, E_6, SO(5, 5), SL(5), SL(3) $\times$ SL(2), SL(2) $\times$ SO(1, 1) for $D = 2, 3, 4, 5, 6, 7, 8, 9$ respectively) is much bigger and contains both the T-duality group [9] (O($10-D$,$10-D$) in D-dimensions) and the S-duality group [10,11] (SL(2, R)) as the subgroup. In this paper, we will make use of the SL(2, R) invariance of type II theory in D-dimensions for $4 \leq D < 10$, to construct the SL(2, Z) multiplets of macroscopic string-like solutions. Since in ten dimensions an SL(2, Z) multiplet of string-like solutions in type IIB theory has already been shown to exist by Schwarz, it is not difficult to understand that such solutions should also exist in $4 \leq D < 10$ by direct dimensional reduction. For $D \leq 8$ there must exist more string-like solutions which should form multiplets of bigger symmetry group of type II theory. Since we are not restricting ourselves to any particular space-time dimensionality, we construct the multiplets of string-like solutions belonging to only a subgroup of this bigger symmetry group. Note that unlike the T-duality group, the SL(2, R) symmetry remains the symmetry group of type II string theory in any $D < 10$. Moreover, since the T-duality group and the S-duality group act differently on

the background fields, they do not commute with each other and therefore it is quite non-trivial to see how they combine to form the bigger symmetry group, namely, the U-duality group. For exapmle, in six dimensions it has been shown by Sen and Vafa [12] how the complete U-duality group SO(5,5; Z) arise as a consequence of the T-duality symmetry O(4,4; Z) and the SL(2, Z) symmetry in type II theory. The transformation rules for the various background fields have also been obtained by non-trivial manipulations. But it should be emphasized that there is no general rule to find the U-duality symmetry and therefore we consider the non-perturbative subgroup of this bigger symmetry group. As in the ten dimensional case the string-like solutions constructed in this paper form SL(2, Z) multiplets as their charges as well as the tensions are found to be given by SL(2, Z) covariant expressions. Since these string states are BPS saturated, this gives a strong evidence that the compactified theories also possess an exact SL(2, Z) invariance. The solutions constructed here will be characterized, as their counterpart in ten dimensions, by two relatively prime integers and will be shown to form a stable spectrum as they are prevented from decaying into other states by the charge conservation and the tension gap relation.

The D-dimensional effective action common to all string theories has the from:

$$\tilde{S}_D = \frac{1}{2\kappa^2} \int d^D x \sqrt{-G} e^{-2\phi} \left(R + 4\partial_\mu \phi \partial^\mu \phi - \frac{1}{12} H^{(1)}_{\mu\nu\lambda} H^{(1)\,\mu\nu\lambda} \right) \tag{1}$$

where $G = (\det G_{\mu\nu})$, $G_{\mu\nu}$ being the string metric, R is the scalar curvature associated with $G_{\mu\nu}$, ϕ is the dilaton and $H^{(1)}_{\mu\nu\lambda}$ is the field strength associated with the Kalb-Ramond antisymmetric tensor field $B^{(1)}_{\mu\nu}$. These are the massless modes which couple to any string theory. By a conformal scaling of the metric

$$G_{\mu\nu} = e^{\frac{4}{D-2}\phi} g_{\mu\nu} \tag{2}$$

we can rewrite the action in the Einstein frame as follows:

$$S_D = \frac{1}{2\kappa^2} \int d^D x \sqrt{-g} \left[R - \frac{4}{D-2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{12} e^{-\frac{8}{D-2}\phi} H^{(1)}_{\mu\nu\lambda} H^{(1)\,\mu\nu\lambda} \right] \tag{3}$$

Here R now represents the scalar curvature associated with the canonical metric $g_{\mu\nu}$. The above D-dimensional action is precisely the one studied by Dabholkar et. al. [13] to construct the macroscopic string-like solutions. (We will show later that the D-dimensional type II action reduces to the action (3) in a particular limit.) Note that we have not

rescaled the dilaton as done in ref.[13]. The solution is given by the following metric and the other field configurations,

$$ds^2 = A^{-\frac{D-4}{D-2}}\left(-dt^2 + (dx^1)^2\right) + A^{\frac{2}{D-2}}\delta_{ij}dx^i dx^j \tag{4}$$

Here i, $j = 2, \ldots, D-1$. A is a function (whose explicit form is given below) of radial coordinate r only, where, $r^2 \equiv \delta_{ij}x^i x^j$. It is clear from (4) that the string is aligned to x^1. The only non-zero component of the antisymmetric tensor field is

$$B^{(1)}_{01} = A^{-1} = e^{2\phi} \tag{5}$$

It should be noted that the expression (5) is formally independent of the dimensionality of the theory and this is the reason for the formal similarity of the string-like solutions in $D = 10$ and $D < 10$ as we will see later. Of course, the information about the dimensionality resides in the function A. The function $A(r)$ is given by,

$$A(r) = \begin{cases} 1 + \frac{Q}{(D-4)r^{D-4}\Omega_{D-3}} & \text{for} \quad D > 4 \\ 1 - \frac{Q}{2\pi}\log r & \text{for} \quad D = 4 \end{cases} \tag{6}$$

Here Q is the 'electric' charge associated with the antisymmetric tensor field $B^{(1)}_{\mu\nu}$ carried by the string and is defined as,

$$Q = \int_{S^{D-3}} * e^{-\frac{8}{D-2}\phi} H^{(1)} \tag{7}$$

Here $*$ denotes the Hodge dual in terms of the canonical metric and the integral is to be evaluated over $(D-3)$-dimensional unit sphere S^{D-3} surrounding the string. Also Ω_{D-3} represents the volume of the unit $(D-3)$-dimensional sphere given by $\Omega_{D-3} = \frac{2(\pi)^{\frac{D-2}{2}}}{\Gamma[\frac{1}{2}(D-2)]}$.

Recall that in ref.[13], the field equations and their solution (6) were obtained by coupling the supergravity action S_D to a macroscopic string source S_σ given by

$$S_\sigma = -\frac{T}{2}\int d^2\sigma \left(\sqrt{\gamma}\gamma^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu} e^{\frac{4}{D-2}\phi} + \epsilon^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu B^{(1)}_{\mu\nu}\right) \tag{8}$$

where T is the string tension and $\gamma_{\alpha\beta}$ is the world-sheet metric. So, the D-dimensional string-like solution in (4) is not 'solitonic' in the strict sense as the metric shows a curvature singularity at $r = 0$ and also the field equation $\nabla^2 A$ has a delta function singularity at that point [11]. By evaluating the 'electric' charge explicitly using Eq.(7) we can derive the relation between the charge and string tension as $Q = 2\kappa^2 T$. We would like to point out that the supergravity action (3) has a manifestly SL(2, R) invariance by which the

fields in (3) can be rotated to convert the action to the D-dimensional type II action including the R-R terms. Since we have already obtained the toroidal compactification of type IIB theory in ref.[8], we will show how starting from D-dimensional type II action we can obtain the action (3) of Dabholkar et. al.

It is well-known that the equations of motion of type IIB supergravity theory can not be obtained from a covariant action [14] because of the presence of a four-form gauge field with the self-dual field strength in the spectrum. This gauge field couples to a self-dual three-brane which can give rise to string solution in $D \leq 8$. But, we are not going to consider this type of string solution and set the corresponding field-strength F_5 to zero. There are also magnetically charged string solution for type II theory in $D \leq 6$, but since we are not restricting ourselves to any particular dimensionality we will not consider those kinds of solutions also. Now as we set $F_5 = 0$, the type IIB equations of motion can be derived from the following covariant action*:

$$\begin{aligned} \tilde{S}_{10}^{\rm IIB} &= \frac{1}{2\kappa^2}\int d^{10}\hat{x}\sqrt{-\hat{G}}\Big[e^{-2\hat{\phi}}\Big(\hat{R} + 4\partial_{\hat{\mu}}\hat{\phi}\partial^{\hat{\mu}}\hat{\phi} - \frac{1}{12}\hat{H}^{(1)}_{\hat{\mu}\hat{\nu}\hat{\lambda}}\hat{H}^{(1)\,\hat{\mu}\hat{\nu}\hat{\lambda}}\Big) \\ &\qquad -\frac{1}{2}\partial_{\hat{\mu}}\hat{\chi}\partial^{\hat{\mu}}\hat{\chi} - \frac{1}{12}\Big(\hat{H}^{(2)}_{\hat{\mu}\hat{\nu}\hat{\lambda}} + \hat{\chi}\hat{H}^{(1)}_{\hat{\mu}\hat{\nu}\hat{\lambda}}\Big)\Big(\hat{H}^{(2)\,\hat{\mu}\hat{\nu}\hat{\lambda}} + \hat{\chi}\hat{H}^{(1)\,\hat{\mu}\hat{\nu}\hat{\lambda}}\Big)\Big] \end{aligned} \tag{9}$$

Here the metric $\hat{G}_{\hat{\mu}\hat{\nu}}$, the dilaton $\hat{\phi}$ and the antisymmetric tensor $\hat{B}^{(1)}_{\hat{\mu}\hat{\nu}}$ (with $\hat{H}^{(1)} = d\hat{B}^{(1)}$) represent the massless modes in the NS-NS sector of type IIB theory. Also the scalar $\hat{\chi}$ and $\hat{B}^{(2)}_{\hat{\mu}\hat{\nu}}$ (with $\hat{H}^{(2)} = d\hat{B}^{(2)}$) represent the massless modes in the R-R sector. We have already studied the dimensional reduction of this action in ref.[8]. The reduced action takes the form:

$$\begin{aligned} &\frac{1}{2\kappa^2}\int d^Dx\sqrt{-G}\Big[e^{-2\phi}\Big(R + 4\partial_\mu\phi\partial^\mu\phi - \frac{1}{4}G_{mn}F^{(3)\,m}_{\mu\nu}F^{(3)\,\mu\nu,\,n} + \frac{1}{4}\partial_\mu G_{mn}\partial^\mu G^{mn} \\ &\qquad -\frac{1}{4}G^{mp}G^{nq}\partial_\mu B^{(1)}_{mn}\partial^\mu B^{(1)}_{pq} - \frac{1}{4}G^{mp}H^{(1)}_{\mu\nu\,m}H^{(1)\,\mu\nu}_{\;p} - \frac{1}{12}H^{(1)}_{\mu\nu\lambda}H^{(1)\,\mu\nu\lambda}\Big) \\ &\qquad -\frac{1}{2}\Delta\partial_\mu\chi\partial^\mu\chi - \frac{1}{4}\Delta G^{mp}G^{nq}\Big(\partial_\mu B^{(2)}_{mn} + \chi\partial_\mu B^{(1)}_{mn}\Big)\Big(\partial^\mu B^{(2)}_{pq} + \chi\partial^\mu B^{(1)}_{pq}\Big) \\ &\qquad -\frac{1}{4}\Delta G^{mp}\Big(H^{(2)}_{\mu\nu\,m} + \chi H^{(1)}_{\mu\nu\,m}\Big)\Big(H^{(2)\,\mu\nu}_{\;p} + \chi H^{(1)\,\mu\nu}_{\;p}\Big) \\ &\qquad -\frac{1}{12}\Delta\Big(H^{(2)}_{\mu\nu\lambda} + \chi H^{(1)}_{\mu\nu\lambda}\Big)\Big(H^{(2)\,\mu\nu\lambda} + \chi H^{(1)\,\mu\nu\lambda}\Big)\Big] \end{aligned} \tag{10}$$

*Here $\mu, \nu, \ldots = 0, 1, \ldots, D-1$ are the space-time indices and $m, n, \ldots = D, \ldots, 9$ are the internal indices. The ten dimensional objects are denoted with a 'hat'. Note that we are using slightly different notation than in ref.[8].

where the definitions and the reduced forms of various gauge fields are:

$$\hat{G}_{\hat{\mu}\hat{\nu}} \quad \longrightarrow \quad \begin{cases} \hat{G}_{mn} &= G_{mn} \\ G_{\mu m} &= \hat{G}_{\mu m} \;=\; A^{(3)\,n}_{\mu} G_{mn} \\ \hat{G}_{\mu\nu} &= G_{\mu\nu} + G_{mn} A^{(3)\,m}_{\mu} A^{(3)\,n}_{\nu} \end{cases} \tag{11}$$

$$\hat{\phi} \quad = \quad \phi + \frac{1}{2}\log\Delta, \quad \text{where} \qquad \Delta^2 = (\det G_{mn}) \tag{12}$$

$$\hat{\chi} \quad = \quad \chi \tag{13}$$

$$\hat{B}^{(i)}_{\hat{\mu}\hat{\nu}} \quad \longrightarrow \quad \begin{cases} B^{(i)}_{\mu m} &= A^{(i)}_{\mu m} \;=\; \hat{B}^{(i)}_{\mu m} + B^{(i)}_{mn} A^{(3)\,n}_{\mu} \\ B^{(i)}_{\mu\nu} &= \hat{B}^{(i)}_{\mu\nu} + A^{(3)\,m}_{\mu} A^{(i)}_{\nu m} - A^{(3)m}_{\nu} A^{(i)}_{\mu m} - A^{(3)m}_{\mu} B^{(i)}_{mn} A^{(3)n}_{\nu} \end{cases} \tag{14}$$

where $i = 1, 2$. The corresponding field-strengths are given below:

$$H^{(i)}_{\mu mn} \quad = \quad \hat{H}^{(i)}_{\mu mn} \;=\; \partial_\mu B^{(i)}_{mn} \tag{15}$$

$$H^{(i)}_{\mu\nu m} \quad = \quad F^{(i)}_{\mu\nu m} - B^{(i)}_{mn} F^{(3)\,n}_{\mu\nu} \tag{16}$$

where $F^{(i)}_{\mu\nu\,m} = \partial_\mu A^{(i)}_{\nu\,m} - \partial_\nu A^{(i)}_{\mu\,m}$ and $F^{(3)\,m}_{\mu\nu} = \partial_\mu A^{(3)\,m}_{\nu} - \partial_\nu A^{(3)\,m}_{\mu}$ and finally,

$$H^{(i)}_{\mu\nu\lambda} \quad = \quad \partial_\mu B^{(i)}_{\nu\lambda} - F^{(3)\,m}_{\mu\nu} A^{(i)}_{\lambda\,m} + \text{cyc. in } \mu\nu\lambda \tag{17}$$

The reduced action (10) was shown in ref.[8] to have an SL(2, R) invariance which can be better understood by rewriting the action in the Einstein frame. The metric in the Einstein frame is related with the string metric as given in Eq.(2). Using this, the action (10) in the Einstein frame takes the following form:

$$\begin{aligned} &\frac{1}{2\kappa^2}\int d^Dx\sqrt{-g}\,\Big[R - \frac{1}{2}\partial_\mu\tilde{\phi}\partial^\mu\tilde{\phi} - \frac{1}{2}e^{2\tilde{\phi}}\partial_\mu\chi\partial^\mu\chi + \frac{1}{8}\partial_\mu\log\bar{\Delta}\partial^\mu\log\bar{\Delta} \\ &\quad + \frac{1}{4}\partial_\mu g_{mn}\partial^\mu g^{mn} - \frac{1}{4}g_{mn}F^{(3)\,m}_{\mu\nu}F^{(3)\,\mu\nu,\,n} - \frac{1}{4}(\bar{\Delta})^{1/2}g^{mp}g^{nq}e^{-\tilde{\phi}}\partial_\mu B^{(1)}_{mn}\partial^\mu B^{(1)}_{pq} \\ &\quad - \frac{1}{4}(\bar{\Delta})^{1/2}g^{mp}g^{nq}e^{\tilde{\phi}}\left(\partial_\mu B^{(2)}_{mn} + \chi\partial_\mu B^{(1)}_{mn}\right)\left(\partial^\mu B^{(2)}_{pq} + \chi\partial^\mu B^{(1)}_{pq}\right) \\ &\quad - \frac{1}{4}(\bar{\Delta})^{1/2}g^{mp}\left\{e^{-\tilde{\phi}}H^{(1)}_{\mu\nu m}H^{(1)\,\mu\nu}_{\;\;p} + e^{\tilde{\phi}}\left(H^{(2)}_{\mu\nu\,m} + \chi H^{(1)}_{\mu\nu\,m}\right)\left(H^{(2)\,\mu\nu}_{\;\;p} + \chi H^{(1)\,\mu\nu}_{\;\;p}\right)\right\} \\ &\quad - \frac{1}{12}(\bar{\Delta})^{1/2}\left\{e^{-\tilde{\phi}}H^{(1)}_{\mu\nu\lambda}H^{(1)\,\mu\nu\lambda} + e^{\tilde{\phi}}\left(H^{(2)}_{\mu\nu\lambda} + \chi H^{(1)}_{\mu\nu\lambda}\right)\left(H^{(2)\,\mu\nu\lambda} + \chi H^{(1)\,\mu\nu\lambda}\right)\right\}\Big] \end{aligned} \tag{18}$$

where we have defined $\tilde{\phi} = \phi + \frac{1}{2}\log\Delta$. Also, $G_{mn} = e^{\frac{4}{D-2}\phi}g_{mn}$ and so, $\Delta = e^{2\frac{(10-D)}{(D-2)}\phi}\bar{\Delta}$ with $(\bar{\Delta})^2 = (\det g_{mn})$. If we define the following SL(2, R) matrix

$$\mathcal{M}_D \;\equiv\; \begin{pmatrix} \chi^2 + e^{-2\tilde{\phi}} & \chi \\ \chi & 1 \end{pmatrix} e^{\tilde{\phi}} \tag{19}$$

then the action (18) can be expressed in the manifestly SL(2, R) invariant form as,

$$\begin{aligned} \frac{1}{2\kappa^2}\int d^Dx\sqrt{-g}\Big[&R+\frac{1}{4}\mathrm{tr}\,\partial_\mu\mathcal{M}_D\partial^\mu\mathcal{M}_D^{-1}+\frac{1}{8}\partial_\mu\log\bar{\Delta}\partial^\mu\log\bar{\Delta}+\frac{1}{4}\partial_\mu g_{mn}\partial^\mu g^{mn}\\ &-\frac{1}{4}g_{mn}F^{(3)\,m}_{\mu\nu}F^{(3)\,\mu\nu,\,n}-\frac{1}{4}(\bar{\Delta})^{1/2}g^{mp}g^{nq}\partial_\mu\mathcal{B}^T_{mn}\mathcal{M}_D\partial^\mu\mathcal{B}_{pq}\\ &-\frac{1}{4}(\bar{\Delta})^{1/2}g^{mp}\mathcal{H}^T_{\mu\nu\,m}\mathcal{M}_D\mathcal{H}^{\mu\nu}_{\ \ \ p}-\frac{1}{12}(\bar{\Delta})^{1/2}\mathcal{H}^T_{\mu\nu\lambda}\mathcal{M}_D\mathcal{H}^{\mu\nu\lambda}\Big] \end{aligned} \tag{20}$$

Here we have defined $\mathcal{B}_{mn} \equiv \begin{pmatrix} B^{(1)}_{mn} \\ B^{(2)}_{mn} \end{pmatrix}$, $\mathcal{H}_{\mu\nu\,m} \equiv \begin{pmatrix} H^{(1)}_{\mu\nu\,m} \\ H^{(2)}_{\mu\nu\,m} \end{pmatrix}$, and $\mathcal{H}_{\mu\nu\lambda} \equiv \begin{pmatrix} H^{(1)}_{\mu\nu\lambda} \\ H^{(2)}_{\mu\nu\lambda} \end{pmatrix}$. The superscript '$T$' denotes the tranpose of a matrix. The action (20) is invariant under the following global SL(2, R) transformation:

$$\begin{aligned} \mathcal{M}_D \ &\rightarrow\ \Lambda\mathcal{M}_D\Lambda^T, \qquad \mathcal{B}_{mn}\ \rightarrow\ (\Lambda^{-1})^T\mathcal{B}_{mn}\\ \begin{pmatrix} A^{(1)}_{\mu\,m} \\ A^{(2)}_{\mu\,m} \end{pmatrix} \equiv \mathcal{A}_{\mu\,m}\ &\rightarrow\ (\Lambda^{-1})^T\mathcal{A}_{\mu\,m}, \qquad \begin{pmatrix} B^{(1)}_{\mu\nu} \\ B^{(2)}_{\mu\nu} \end{pmatrix} \equiv \mathcal{B}_{\mu\nu}\ \rightarrow\ (\Lambda^{-1})^T\mathcal{B}_{\mu\nu}\\ g_{\mu\nu}\ &\rightarrow\ g_{\mu\nu}, \qquad g_{mn}\ \rightarrow\ g_{mn}, \qquad \text{and}\quad A^{(3)\,m}_\mu\ \rightarrow\ A^{(3)\,m}_\mu \end{aligned} \tag{21}$$

where $\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the SL(2,R) transformation matrix and a, b, c, d are constants satisfying $ad - bc = 1$. Now if we set $G_{mn} = \delta_{mn}$, $\Delta = 1$, $A^{(3)\,m}_\mu = A^{(i)}_{\mu\,n} = B^{(i)}_{mn} = 0$, then the action (20) reduces to:

$$\begin{aligned} \frac{1}{2\kappa^2}\int d^Dx\sqrt{-g}\Big[&R+\frac{1}{4}\mathrm{tr}\,\partial_\mu\mathcal{M}_D\partial^\mu\mathcal{M}_D^{-1}+\frac{1}{8}\partial_\mu\log\bar{\Delta}\partial^\mu\log\bar{\Delta}\\ &+\frac{1}{4}\partial_\mu g_{mn}\partial^\mu g^{mn}-\frac{1}{12}(\bar{\Delta})^{1/2}\mathcal{H}^T_{\mu\nu\lambda}\mathcal{M}_D\mathcal{H}^{\mu\nu\lambda}\Big] \end{aligned} \tag{22}$$

This action is SL(2, R) invariant under the transformation (21). Note that both g_{mn} and $\bar{\Delta}$ are SL(2, R) invariant. Also, $\mathcal{M}_D$ in (21) is as given in (19) with $\tilde{\phi}$ replaced by ϕ, the D-dimensional dilaton as $\Delta = 1$ in this case. Note also that although we have set $G_{mn} = \delta_{mn}$ and $\Delta = 1$, but as they are not SL(2, R) invariant, non-trivial values of G_{mn} and Δ will be generated through SL(2, R) transformation. It can be easily checked that the SL(2, R) invariant action (22) gets precisely converted to the effective action (3) considered by Dabholkar et. al. by setting the R-R fields to zero. Thus, we note that the action (3) is a special case of the more general type II action (22) and so the solution (6) is a particular case of a general solution that we are going to construct.

Our strategy to construct the string-like solution for type II theory in D-dimensions is to start with the action (3) and the solution (4)–(6) and then use the SL(2, R) symmetry to rotate the solution corresponding to the full type II theory. Since the starting solution

has only $B^{(1)}_{\mu\nu}$ field and the corresponding charge, the final solution will have both the fields $B^{(1)}_{\mu\nu}$ and $B^{(2)}_{\mu\nu}$ and their charges. In order to describe the complete string solution we also have to specify the asymptotic values of both ϕ and χ as $r \to \infty$. Under the transformation $\mathcal{M}_D \to \Lambda \mathcal{M}_D \Lambda^T$ and $\mathcal{B}_{\mu\nu} \to (\Lambda^{-1})^T \mathcal{B}_{\mu\nu}$, the complex scalar field $\lambda = \chi + ie^{-\phi}$ and the $\mathcal{B}^{(i)}_{\mu\nu}$ transform as,

$$\lambda \quad \to \quad \frac{a\lambda + b}{c\lambda + d} \tag{23}$$

$$\begin{aligned} B^{(1)}_{01} &\to dB^{(1)}_{01} - cB^{(2)}_{01} \\ B^{(2)}_{01} &\to -bB^{(1)}_{01} + aB^{(2)}_{01} \end{aligned} \tag{24}$$

We first construct the solution corresponding to the simplest choice of $\lambda_0 = i$ (i.e. for $\chi_0 = \phi_0 = 0$) as in ref.[4]. Here subscript '0' represents the asymptotic value. We also replace the 'electric' charge Q in (6) by $\alpha_{(q_1,q_2)} = \sqrt{q_1^2 + q_2^2}Q$. From the relation between charge and string tension $Q = 2\kappa^2 T$, it is clear that T also has to be replaced by $T_{(q_1,q_2)} = \sqrt{q_1^2 + q_2^2}T$. Using the value of $B^{(1)}_{01} = A^{-1}_{(q_1,q_2)}$, where $A_{(q_1,q_2)}$ is as given in (6) with Q replaced by $\alpha_{(q_1,q_2)}$, we can easily calculate the 'electric' charges associated with the transformed fields $B^{(1)}_{01}$ and $B^{(2)}_{01}$ to be $a\sqrt{q_1^2 + q_2^2}Q$ and $c\sqrt{q_1^2 + q_2^2}Q$. Demanding that the charges be quantized, the constants a, b, c, d get completely fixed as follows:

$$\Lambda = \frac{1}{\sqrt{q_1^2 + q_2^2}} \begin{pmatrix} q_1 & -q_2 \\ q_2 & q_1 \end{pmatrix} \tag{25}$$

where q_1 and q_2 are integers. With this Λ, $B^{(i)}_{01}$ and λ are given as

$$B^{(i)}_{01} = \frac{q_i}{\sqrt{q_1^2 + q_2^2}} A^{-1}_{(q_1,q_2)} \tag{26}$$

$$\lambda = \frac{iq_1 A^{1/2}_{(q_1,q_2)} - q_2}{iq_2 A^{1/2}_{(q_1,q_2)} + q_1} = \frac{q_1 q_2 \left(A_{(q_1,q_2)} - 1\right) + i\,(q_1^2 + q_2^2)\,A^{1/2}_{(q_1,q_2)}}{q_1^2 + q_2^2 A_{(q_1,q_2)}} \tag{27}$$

Note that the solution formally has the same form as the ten dimensional solution [4]. Asymptotically for $D > 4$ as $r \to \infty$, $A_{(q_1,q_2)} \to 1$ and so, $\lambda \to i$. Note from Eqs.(4) and (6), that the metric has a nice asymptotic limit for $D > 4$, but for $D = 4$ the metric has logarithmic divergence. However, it was shown in ref.[13] that one can still write the metric in a "weak-field" type expansion and calculate the total energy density in the ambient space. It was found that the energy density outside the string core ($r = 0$) vanishes reflecting the fact that effectively the function $A_{(q_1,q_2)}$ tends to one as we go away

from the string core. It should be mentioned that although originally $\chi = 0$, a non-trivial χ is generated through SL(2, R) transformation as can be seen from (27). Similarly, although we started with a trivial internal metric $G_{mn} = \delta_{mn}$ and $\Delta = 1$, a non-trivial G_{mn} and Δ will be generated after the SL(2, R) transformation as given below:

$$
\begin{aligned}
G_{mn} &= \left(\frac{q_1^2 + q_2^2 A_{(q_1,q_2)}}{q_1^2 + q_2^2}\right)^{4/(D-2)} \delta_{mn} \\
\Delta &= \left(\frac{q_1^2 + q_2^2 A_{(q_1,q_2)}}{q_1^2 + q_2^2}\right)^{2(10-D)/(D-2)}
\end{aligned} \tag{28}
$$

In deriving (28) we have used the transformation rule of $e^{-\phi}$ obtained in (27) and the fact that $g_{mn} = e^{-\frac{4}{D-2}\phi} G_{mn}$ is invariant under SL(2, R) transformation.

We next generalize the construction for arbitrary vacuum modulus λ_0. So, we start with an arbitrary value of the charge $\alpha_{(q_1,q_2)} = \Delta^{1/2}_{(q_1,q_2)} Q$ in (6) and choose,

$$
\begin{aligned}
\Lambda = \Lambda_1 \Lambda_2 &= \begin{pmatrix} e^{-\phi_0/2} & \chi_0 e^{\phi_0/2} \\ 0 & e^{\phi_0/2} \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \\
&= \begin{pmatrix} e^{-\phi_0} \cos\theta + \chi_0 \sin\theta & -e^{-\phi_0} \sin\theta + \chi_0 \cos\theta \\ \sin\theta & \cos\theta \end{pmatrix} e^{\phi_0/2}
\end{aligned} \tag{29}
$$

Note that Λ_2 is the most general SL(2, R) matrix which preserves the vacuum modulus $\lambda_0 = i$ and Λ_1 is the SL(2, R) matrix which transforms it to an arbitrary value $\lambda = \lambda_0$. We point out that the charge quantization condition in the previous case fixed the value of $\cos\theta$ and $\sin\theta$ to have the particular form given in (25). In the present case as the matrix Λ is different, charge quantization condition will yield different values of $\cos\theta$ and $\sin\theta$. Using (29), we find the 'electric' charges associated with $B^{(1)}_{01}$ and $B^{(2)}_{01}$ as,

$$
\begin{aligned}
Q^{(1)} &= \left(e^{-\phi_0/2} \cos\theta + \chi_0 e^{\phi_0/2} \sin\theta\right) \Delta^{1/2}_{(q_1,q_2)} Q \\
Q^{(2)} &= e^{\phi_0/2} \sin\theta \Delta^{1/2}_{(q_1,q_2)} Q
\end{aligned} \tag{30}
$$

By demanding that the charges be quantized, we get from (30),

$$
\begin{aligned}
\sin\theta &= e^{-\phi_0/2} \Delta^{-1/2}_{(q_1,q_2)} q_2 \\
\cos\theta &= e^{\phi_0/2} \left(q_1 - q_2 \chi_0\right) \Delta^{-1/2}_{(q_1,q_2)}
\end{aligned} \tag{31}
$$

where q_1 and q_2 are integers. Using $\cos^2\theta + \sin^2\theta = 1$ we obtain from (31), the value of $\Delta_{(q_1,q_2)}$ to be

$$
\begin{aligned}
\Delta_{(q_1,q_2)} &= e^{-\phi_0} q_2^2 + \left(q_1 - q_2 \chi_0\right)^2 e^{\phi_0} \\
&= (q_1,\ q_2)\, \mathcal{M}^{-1}_{D0} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}
\end{aligned} \tag{32}
$$

where $\mathcal{M}_{D0} = \begin{pmatrix} \chi_0^2 + e^{-2\phi_0} & \chi_0 \\ \chi_0 & 1 \end{pmatrix} e^{\phi_0}$. The important point to note here is that the expression for $\Delta_{(q_1,q_2)}$ is SL(2, R) covariant† and so is the charge as well as the tension of a general string. This can be easily seen as the charge $\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$ transforms as $\Lambda \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$ under SL(2, R) transformation. With the value of $\sin\theta$ and $\cos\theta$ in (31) the transformed antisymmetric tensor field components can be obtained from (21) as,

$$\begin{aligned} B_{01}^{(1)} &= e^{\phi_0} (q_1 - q_2\chi_0)\, \Delta_{(q_1,q_2)}^{-1/2} A_{(q_1,q_2)}^{-1} \\ B_{01}^{(2)} &= e^{\phi_0} \left(q_2|\lambda_0|^2 - q_1\chi_0\right) \Delta_{(q_1,q_2)}^{-1/2} A_{(q_1,q_2)}^{-1} \end{aligned} \tag{33}$$

which can be written compactly as follows,

$$\begin{pmatrix} B_{01}^{(1)} \\ B_{01}^{(2)} \end{pmatrix} = \mathcal{M}_{D0}^{-1} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \Delta_{(q_1,q_2)}^{-1/2} A_{(q_1,q_2)}^{-1} \tag{34}$$

It can be checked that both sides of (34) has the right transformation property under SL(2, R). Using (31), λ can be calculated as follows:

$$\begin{aligned} \lambda &= \frac{q_1\chi_0 - q_2|\lambda_0|^2 + iq_1 e^{-\phi_0} A_{(q_1,q_2)}^{1/2}}{q_1 - q_2\chi_0 + iq_2 e^{-\phi_0} A_{(q_1,q_2)}^{1/2}} \\ &= \frac{\chi_0 e^{-\phi_0}\Delta_{(q_1,q_2)} + q_1q_2 e^{-2\phi_0}\left(A_{(q_1,q_2)} - 1\right) + i\Delta_{(q_1,q_2)} A_{(q_1,q_2)}^{1/2} e^{-2\phi_0}}{e^{-\phi_0}\Delta_{(q_1,q_2)} + q_2^2 e^{-2\phi_0}\left(A_{(q_1,q_2)} - 1\right)} \end{aligned} \tag{35}$$

So, from (35) we find the transformed value of G_{mn} and Δ as

$$\begin{aligned} G_{mn} &= \left(\frac{e^{-\phi_0}\Delta_{(q_1,q_2)} + q_2^2 e^{-2\phi_0}\left(A_{(q_1,q_2)} - 1\right)}{\Delta_{(q_1,q_2)} e^{-\phi_0}}\right)^{4/(D-2)} \delta_{mn} \\ \Delta &= \left(\frac{e^{-\phi_0}\Delta_{(q_1,q_2)} + q_2^2 e^{-2\phi_0}\left(A_{(q_1,q_2)} - 1\right)}{\Delta_{(q_1,q_2)} e^{-\phi_0}}\right)^{2(10-D)/(D-2)} \end{aligned} \tag{36}$$

Note that the generation of non-trivial Δ is consistent with the SL(2, R) invariance of type II string effective action because the matrix $\mathcal{M}_D$ in Eq.(19) constructed to show the invariance in fact involves $\tilde{\phi} = \phi + \frac{1}{2}\log\Delta$.

So, starting from the macroscopic string-like solution of Dabholkar et. al. [13] in any $D < 10$, we have been able to construct the SL(2, Z) multiplets of string-like solutions

†As these string states are BPS saturated, their tensions will not be renormalized and thus the SL(2, Z) covariant formula of string tension gives a strong evidence for the conjectured SL(2, Z) invariance in the theory.

of type II string theory given by the metric in Eq.(4) and other field configurations in Eqs.(33)– (36). This generalizes the construction of SL(2, Z) multiplets of string solutions in $D = 10$ by Schwarz [4] to any $D < 10$. Although formally the expressions (33)–(35) look identical to the ten dimensional solutions of Schwarz, but they are completely different. This is because the function $A_{(q_1,q_2)}$ is totally different for different D as can be seen from Eq.(6). The solutions as we have seen are characterized by two integers (q_1, q_2) corresponding to the charges associated with $B^{(1)}_{01}$ and $B^{(2)}_{01}$.

Let us now discuss the stability of these solutions [6]. We have mentioned earlier that the charge and the tension of a general (q_1, q_2) string is given by SL(2, Z) covariant expressions,

$$\alpha_{(q_1,q_2)} = \Delta^{1/2}_{(q_1,q_2)} Q = \sqrt{e^{-\phi_0} q_2^2 + (q_1 - q_2\chi_0)^2\, e^{\phi_0}}\, Q \tag{37}$$

$$T_{(q_1,q_2)} = \Delta^{1/2}_{(q_1,q_2)} T = \sqrt{e^{-\phi_0} q_2^2 + (q_1 - q_2\chi_0)^2\, e^{\phi_0}}\, T \tag{38}$$

Using (37) and (38), it is easy to check that when $\chi = 0$, $(\alpha_{(p_1,p_2)} + \alpha_{(q_1,q_2)})^2 \geq \alpha^2_{(p_1+q_1,p_2+q_2)}$ and similarly, $(T_{(p_1,p_2)} + T_{(q_1,q_2)})^2 \geq T^2_{(p_1+q_1,p_2+q_2)}$. Since both $\alpha_{(q_1,q_2)}$ and $T_{(q_1,q_2)}$ are positive real numbers, we conclude,

$$\alpha_{(p_1,p_2)} + \alpha_{(q_1,q_2)} \geq \alpha_{(p_1+q_1,p_2+q_2)} \tag{39}$$

$$T_{(p_1,p_2)} + T_{(q_1,q_2)} \geq T_{(p_1+q_1,p_2+q_2)} \tag{40}$$

The equality holds when $p_1 q_2 = p_2 q_1$ or in other words when $p_1 = nq_1$ and $p_2 = nq_2$ with n being an integer. So, when q_1, q_2 are relatively prime, the inequality prevents the string from decaying into multiple string states, as this configuration is energetically more favorable than others. Eq.(40) is what we have called the tension gap equation. Also, from (39) we note that when q_1, q_2 are relatively prime, the charge conservation can not be satisfied if the string breaks up into multiple strings. Thus the configuration (q_1, q_2), with q_1 and q_2 relatively prime, is perfectly stable and denotes a bound state configuration [15] of q_1 fundamental strings with q_2 D-strings [16].

To conclude, we have constructed in this paper the SL(2, Z) multiplets of macroscopic string-like solutions of type II theory in any $D < 10$. This construction is made possible by a recent observation of the SL(2, R) invariance of toroidally compactified type IIB string effective action. This generalizes the construction of SL(2, Z) multiplets of string-like solutions of type IIB string theory in $D = 10$ by Schwarz. Our solutions have formal similarity with the solutions in $D = 10$, but they are totally different as they involve

dimensionally dependent functions. The string-like solutions in $D < 10$ are also characterized by two relatively prime integers, as their counterpart in $D = 10$, corresponding to the charges of two antisymmetric tensor fields in the theory. We have also discussed in brief the stability of the solutions from the charge conservation and tension gap relation. As we have mentioned earlier, there are more string-like solutions not only with electric charge but also with magnetic charge in type II theories in lower dimensions which should form multiplets of bigger symmetry group, the U-duality group. Apart from the string-like solutions, there are also other p-brane solutions in these theories which deserve a systematic study to properly identify the complete U-duality group. This will provide strong evidence for the conjecture of the U-duality symmetries in those theories.

Acknowledgements:

I am grateful to John Schwarz for very helpful correspondences. I would also like to thank Jnanadeva Maharana for useful discussions.

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